Functions on path space and applications

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- Let $V \cong \mathbb{R}^d$ with $d \geq 1$.
- I will talk about a rough path perspective to real-valued functions on paths:

$$C([0, T]; V) \to \mathbb{R}.$$

Applications: machine learning

Applications: finance

- Financial derivatives are essentially functions $F : C([0,T]; V) \to \mathbb{R}$.
- A classical problem is to find the *price* of the financial derivative F, which is given by

$$\mathbb{E}^{\mathbb{Q}}[F(X)]$$

for some probability measure $\mathbb{Q}$.

Definition (Extended tensor algebra)

The extended tensor algebra over V, denoted by $T((V))$, is defined by

$$T((V)) := \{\mathrm{a} = (a_0, a_1, \ldots, a_k, \ldots) : a_k \in V^{\otimes k} \text{ for each } k \in \mathbb{N}\}.$$

It is an algebra with the sum $+$ and the product $\otimes$.

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Definition (Truncated tensor algebra)

Let $n \geq 1$. The truncated tensor algebra of order n over V is the subalgebra

$$T^n(V) := \bigoplus_{k=0}^{n} V^{\otimes k} \hookrightarrow T((V)).$$

Definition (Signature of a path)

Let $X : [0, T] \to V$ be a continuous path such that the integrations below make sense. We define the *signature* of X by

$$\mathbb{X}^{<\infty}_{s,t} := (1, \mathbb{X}^1_{s,t}, \ldots, \mathbb{X}^k_{s,t}, \ldots) \in T((V)) \quad \text{for } 0 \le s \le t \le T,$$

where

$$\mathbb{X}^k_{s,t} := \int_{s<u_1<\ldots<u_k<t} dX_{u_1} \otimes \cdots \otimes dX_{u_k} \in V^{\otimes k}.$$

The *truncated signature* of order n is defined by

$$\mathbb{X}^{\le n}_{s,t} := (1, \mathbb{X}^1_{s,t}, \ldots, \mathbb{X}^n_{s,t}) \in T^n(V).$$

- If X has bounded variation, the integrals can be understood in the sense of Riemann–Stieltjes.
- If X is a semimartingale, we can define the integrals in the sense of Itô or Stratonovich.

Example

For paths of bounded variation,

- $\mathbb{X}^1_{0,T}$ is just the increment of the path, $X_T - X_0 \in V$.

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- $\mathbb{X}^1_{0,T}$ is just the increment of the path, $X_T - X_0 \in V$.
- $\mathbb{X}^2_{0,T} \in V^{\otimes 2}$ is the Lévy area of X.
- Higher order terms of the signature capture other features of the trajectory of X.

Definition (Geometric p−rough paths)

Let $1 \leq p < \infty$. Denote by $G\Omega_p([0,T];V)$ the closure (on a certain metric space) of the truncated signatures of order $\lfloor p \rfloor$ of paths with bounded variation.

Definition (Geometric $p-$rough paths)

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Denote by $\widetilde{G\Omega_p}([0, T]; V) \subset G\Omega_p([0, T]; \mathbb{R} \times V)$ the closure of the truncated signatures of order $\lfloor p \rfloor$ of the paired paths (t, X_t), with $X : [0, T] \to V$ a continuous path of bounded variation.

- The signature of a semimartingale in the sense of Stratonovich is a geometric rough path.
- The signature defined in the sense of Itô, however, is not a geometric rough path.

- The full signature on $[0, T]$ of a geometric rough path fully characterises the path up to *tree-like equivalences* (Boedihardjo et al., 2016).

Rough paths

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- Linear functions on signatures form an algebra (Lyons et al., 2004).
- Hence, by Stone–Weierstrass, linear functions on signatures are dense on continuous functions ((Fawcett, 2002), (Levin et al., 2016)).

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- G induces a continuous function $F : \widetilde{G\Omega_p}([0,T];V) \to \mathbb{R}$.
- Hence, we will consider continuous functions on $\widetilde{G\Omega_p}$.

Theorem (Density of linear functions on the signature)

Let $F : \mathcal{K} \to \mathbb{R}$ be continuous, where $\mathcal{K} \subset \widetilde{G\Omega_p}([0,T];V)$ is compact. Let $\varepsilon > 0$. Then, there exists $\ell \in T((\mathbb{R} \times V))^*$ such that

$$|F(\mathbb{X}) - \langle \ell, \mathbb{X}^{<\infty}_{0,T} \rangle| < \varepsilon \quad \forall\, \mathbb{X} \in \mathcal{K}.$$

Signatures and functions on paths

- Signatures transform nonlinear relationships into linear ones.
- Hence, signatures convert the problem of dealing with complex and possibly unknown nonlinear functions on paths into a linear regression problem.
- This has been leveraged on applications in machine learning, finance, etc.

Thank you!